Numbers

Question: What is mathematics?!

My own understanding of maths is that it is like a language. It is used to represent thoughts and ideas that are about physical quantities. Just as language, once invented, gives rise to its own structures like poetry that create new expressions that were not there in the first place, maths also, once invented in the form of basic numbers, then gives rise to its own structures - arithmetic, algebra etc. that allow us to think of more advanced ideas. But thinking of maths as a collection of symbols, just like a language, is useful and we can use it as a theme through this course.

Perhaps this is too broad a question for the first class. So let us take a slightly more manageable version of this question - **what is mathematics used for?**

Trade and exchange are at the heart of economics, and before you trade you need to decide how much to trade, and for that you need to measure what you are trading. Hence measurement is fundamental for economics. It was also fundamental for societies that were first settling down as a result of a move from hunter-gatherer to agricultural societies. Hence we see the emergence of sophisticated systems of numbers and counting in ancient societies [Egyptian, Babylonian, Mayan etc.]

Q. What are numbers?

If you are in a place where you don't know the language and you wanted to buy two bottles of water, how would you manage?

You are representing one bottle with one finger. And this can be for anything - one banana, one person, one hour etc. As we can run out of fingers pretty fast, so other symbols are needed to denote higher quantities.

One can use lines (tally system) but that also becomes unmanageable beyond a certain point.

Egyptian numbers

Mayan numbers

Mayan is a base twenty system - which means you have twenty different symbols for numbers zero to nineteen and then you denote higher numbers by using the same symbols to represent multiples of twenty, 400 and so on.

The widely used Indo-arabic number system is base ten - it has symbols for numbers zero to nine then the same numbers denote higher multiples of ten, hundred and so on. For computers, a number system with base two, known as binary is used, because the two symbols needed for zero and one, can be represented by the absence or presence of electricity, or off and on positions of a switch.

Let us construct a base four number system. Let us use the usual
Zero: 0
One: 1
Two: 2
Three: 3

What would be the following numbers?
Five: 11
Nine: 21
Nineteen: 43

Can you now construct a base five number system?
Can you write the following numbers in this system?
i) Five ii) Nine iii) Nineteen iv) Twenty three v)Twenty five
[Answers: 10, 14, 34, 43, 100]

This is just to show you that the way we represent numbers - 5, 15, 200 etc. is just one of the ways. There is nothing special about it except that it has become widely used because of accidents of history. In another world (multiverse!) there could be a different number system being or the same system with different symbols for the numbers.

These numbers are called **Natural numbers** and were used for trade and for keeping count of items. Here is where the operations of addition comes in. The idea of **addition** is straightforward: you have some quantity of an item, and then you add more quantity of the same item. Now as societies introduced symbols for numbers, some of them also introduced symbols for this idea of adding the same quantity. The one we commonly use is '+', but there are other ways, sometimes more efficient ones. For example in code (or in Excel) we sometimes use sum(1,4,9,10) instead of 1+4+9+10. They both mean the same thing, the language is a little different!

Addition has a property called being '**commutative**' (no need to remember the word), which means that the order of numbers doesn't matter. If you add 2 to 3, or 3 to 2, you get the same outcome, or 2+3=3+2=5. This is intuitively obvious if we think in terms of counting.

Now if I am an ancient store keeper sitting at the construction site of the Pyramids, keeping a stock of how many stone slabs were brought in, I will keep adding. But then some stone slabs will also be taken out, hence I need to **remove** some numbers from my count and this is nothing but subtraction, which has its own symbol. But the interesting thing is what happens if I reverse the order of subtraction.

I had five slabs and two were taken away, so I am left with three, i.e. 5-2=3. But what if I do the opposite? What is 2-5? One answer is that I cannot do it. If I have two slabs then five slabs cannot be taken. Another answer is that if my records show that there were two slabs and then five were taken, then I must have missed adding a least three slabs somewhere! This idea of a missing quantity, or a quantity that needs to be replenished to balance things out is represented as a **negative number.** In other words a (-3) indicates the absence of a (+3), i.e. 2-5**+3**=0.

So, now with subtraction, we have a new kind of number that represents the absence of things, as opposed to the usual numbers we discussed above that represent the presence of things.

Natural numbers and negative numbers are together known as Integers [comes from the same latin root as ‘intact’ or ‘entire’ - basically means no fractions].

Now let us imagine an ancient farmer wants to trade. **The farmer wants to trade the wheat grown by them for fish.** The fish can be counted given the number system that is used in their society, but what about wheat - are they supposed to count every grain?!!

As opposed to the number of fish, one cannot count the weight on one's fingers or use the numbers. Hence there is a fundamental difference in the type of quantity being measured - countable quantities such as fish, fruit, etc are called **discrete,** and those that cannot be counted are called **continuous.**

Give examples of discrete and continuous

Now the question is how to measure continuous quantities like weights, lengths, times etc. **Just like we use one discrete quantity - fingers, to represent all kinds of discrete quantities, we can do the same - what continuous quantity is used to represent others?**

A line drawn on the board can represent length, but also weight or time. **But how much length of this line represents how much of weight wheat?**

We do not have a readymade comparison as we had with discrete numbers [one finger represents one bottle]. Hence we need to create an artificial way of comparison called a **unit**.

So we can select an arbitrary weight of wheat and call it one unit [one gram one kg etc], and similarly cut one arbitrary length of line and call it one unit. Note that I can also use an arbitrary unit of length on the board to represent a different unit of length in real life - maps do this all the time.

Now that we have the unit, we can use the existing number system to denote multiples of the unit - 2 unit, 3 unit and so on. But because this is a continuous quantity, when I weight my wheat I may end up with something between 2 and 3 multiples of the unit - how do I represent a quantity that is not an exact multiple of the unit? [Answer: **fractions]**

Suppose it is half of a unit more than two. Then we say 2 ½, the fraction denoting that you divide the unit into two parts (the denominator) and take one (numerator). Or it could be ⅔ and so on.

Now, here is a good example of how the structure we created - of **place based number system**, allows us an elegant way of representing these fractions [mathematicians, like poets, are very fond of elegance!]. As we move left, the place value increases from 1 to 10 to 100. Or, in other words, as we move right, it decreases from 100 to 10 to 1. [Demonstrate with an example], i.e. it gets divided by 10 at each stage. So, if we move even further to the right it should become 1/10, 1/100 and so on, which are fractions. We use the decimal point to denote where the place value is 1 and then we can move freely to the left or right and write down fractions without writing fractions!

The same idea was implemented in measurement through the metric system!

So now we have two divisions- positive and negative numbers that represent the presence or absence of quantities, and discrete and continuous numbers that represent countable and non-countable quantities. We can represent them all visually with a slight modification of the line we used for continuous numbers.

The line now is just not a length but has direction. You always start from 0, to the right are positive numbers - and moving to the right is addition. If you move in steps it is discrete, otherwise it is continuous. To the left of zero are negative numbers, and moving left is subtraction. [Show these operations]

This is called the real number line. **Real** is a term that got coined in a different context, some of you may remember real vs imaginary numbers, but we are not using it in that context. We are just using it to talk about all these kinds of numbers - positive, negative, discrete, continuous, that can be depicted on this line. Any number that can be depicted on the real number line is called a `real' number. In economics, we will almost always deal with quantities that can be measured as a real number - money, output, employment, inflation, all are real numbers. All of them can be depicted on this real number line.

Multiplication and area

Another very important quantity to be measured in ancient and modern societies is **land**! People trying to measure how big an area of land is very quickly realised that they could relate it to the measure of length. Defining a unit of area as the area of a square piece of land with the sides equal to the unit of length proves to be a particularly useful way.
If you have a rectangular piece of land say 5 units wide and 3 units long, then you can divide it into three rows, each containing five unit squares. Hence to get the total number of squares you need to add five three time, or do 5+5+5. This idea of adding the same quantity **multiple** times was called multiplication. This also has various symbols, including * . So, 5+5+5 is 5*3. Now the interesting thing is that if you rotate the rectangle by 90 degrees, it still has the same area. Hence, we know that multiplication is commutative.

The roots of multiplication in area calculation is so deep that even now if a number is multiplied by itself, we call it squaring.

Think of the formula $(a+b)^2=a^2+2ab+b^2$.

We will come to algebra later, but for now let us take some values and see visually how this works out.

[Visual proof done in class]

Now, let us say you are standing in ancient Delhi and I ask you the direction to Patna, what would you say? [Something like go east-southeast, and travel on horseback for x number of days]. You actually need to tell me **two** quantities - the direction, and the distance. Today we measure location with two slightly different quantities - latitude and longitude. Hence some quantities, are actually a combination of two quantities, that are each a real number by themselves. There are other applications where we need to combine even more than two quantities - we will come to the system to deal with that, known as vectors and matrices, later in the course.