Algebra

We now have a system of numbers and can do various operations on them. Often we need to ask a question about a quantity that is related to these numbers and operations. For example, if you need to travel 14 km to get to your destination, and have already travelled 9km, how many kms are left to travel. In terms of the numbers and operations, we are asking what is it that when added to 9 will give us 14,

9 + **?** = 14

Given the nature of addition, we know that we can obtain the unknown quantity by subtracting 9 from 14. Or, more logically, if adding ? km to 9 km would give me 14 km, then if I remove 9 from 14, I should be left with ?.

? = 14 - 9 = 5

Or, if one were inclined to use mathematical logic, one can say that if two quantities are equal then whatever we get by subtracting the same amount from them must also be equal, hence:

9 + **?** - 9 = 14 - 9

And hence, **?** = 5.

Similarly, one can ask if a long roll of cloth is 2 m wide, what length of it should I cut so that I have a cloth of area 3 sq m. Knowing that area of a rectangular shape is given by multiplying the width and length, we can translate this question using numbers and operators into:

2 x **?** = 3

We can use similar logic as the previous situation to say that the unknown quantity can be obtained by dividing 3 by 2 [either by the definition of multiplication or by dividing both sides of the equation by 2].

? = 3/2 = 1.5

It is evident that this is very useful. In economics we can ask similar questions, for example at what rate should India's economy grow if it has to reach the level that China is at today. Again we can use simple operations to find the missing number.

The use of a question mark to signify the missing number is intuitive and uncomplicated, but it becomes unusable as soon as more than one number is missing. For example, if I want to find the principal and the rate of simple annual interest that gives me a total amount owed of Rs 1000 at the end of the year, writing the problem in the following way would be very confusing:

? x (1+?) = 1000

Hence, we would like to substitute the two missing quantities with two different symbols so that we can distinguish between the two. We could say

p x (1+i) = 1000

When letters are used to represent numbers, they are called variables. And mathematics with the use of variables is broadly called algebra. The word comes from arabic 'al-jabr' which was part of the title of a book by Persian mathematician al-Khwarizmi which uses these techniques.

You can see that using letters to represent missing quantities can cause confusion as the symbol for multiplication resembles the letter x. So, very often the multiplication symbol is dropped and when two letters, or a number or a letter are next to each other, it indicates multiplication. [In fact the same problem arises when writing code for computers. There the solution was to replace x with * as the symbol for multiplication.]

$p(1+i) = 1000$

It is clear that this system will work only when there are mostly letters i.e. 2 x a = 2a, but will be very confusing with numbers: 2 x 2 = 2 2 !

[You should also be able to see that there is no unique answer i.e. there is no one combination of p and i that would satisfy the equation - there could be many such as p=800; i=0.25, or p=1000; i=0 . We will come back to this later.]

Useful manipulations:
All the manipulations that we do with variables are derived from what we do with numbers in arithmetic, but some of them are especially useful in algebra:

(1) 'Moving' terms from one side of the equal sign to another: In the examples discussed above, we use this idea of moving terms to find the missing value. When we move a term additively, it changes sign. When we move it multiplicatively, it moves from numerator to denominator and vice versa. If this is confusing, a useful way to think of is the addition or subtraction of the same term on both sides, or the multiplication/division by the same term on both sides (refer to the first two examples in this lecture)

(2) Dealing with brackets:
Multiplication has what is called the **distributive property**. You can visualise this using the example of area : 2x3 + 4x3 = (2+4)x3. We use the same property in algebra, but with variables.
To extend the cloth example above: I have one roll of width 2m and another of width 1m. I need to cut the same length from both rolls but get a total area of 6 sq m. If we use x to denote the length cut, can you write the equation to denote this situation:

$2x + 1x = 6$
or, $(2+1)x=6$
or, $3x=6$
or, $x=6/3=2$

Question: The fixed cost of setting up a factory is Rs 2 crore. The variable cost of producing each unit of good is Rs 100. The capacity of the factory is to produce 5 lakh units in a year. What should I set the price of the good so that I break even by the end of the year?

$500{,}000p$ - $(500{,}100 \times 100)$ - $200{,}00{,}000=0$
or, $500{,}000(p-100)=2{,}00{,}00{,}000$
or, $p-100=2{,}00{,}00{,}000/5{,}00{,}000=40$
or, $p=40+100=140$

(3) Cancelling terms: This is the reverse of adding or multiplying both sides with the same terms. Here, if the same term is already there as an additive or multiplicative term on both sides, then it can be removed or 'cancelled'.

Question: Let us consider the question of economic growth. The pandemic shrunk the GDP of a particular country by 10% during financial year 2020-21. What should be the growth rate in year 2021-22 for the GDP to come back to its original level?

GDP before pandemic: a
GDP at the end of 2020-21: $0.9a$
Rate of growth in 2021-22: x

$0.9a(1+x/100)=a$
or, $0.9\cancel{a}(1+x/100)=\cancel{a}$
or, $1+x/100=1/0.9=1.11$ (approx)
or, $x/100=1.11-1=0.11$
or, $x=11$

Hence, the economy needs to grow by 11%.

Note that the cancellation of a multiplicative term, like here, implies division by that term on both sides. What we did here in step 2 was to divide by a on both sides. Hence, we can only do this if a is not equal to 0.

Proofs and formulas:

Proofs are one of the most important outcomes of mathematics. They show some results that always hold, and are very useful. Hence, to prove something it is not sufficient to show that it holds for a particular set of values - that is just an example. So, if we have to prove that $(a+b)^2=a^2+2ab+b^2$, then we cannot just say let $a=2$ *and* $b=3$, then say LHS$=5^2=25$, and

RHS=4+12+9=25, and hence this is true. We have to show that this will *always* be true. We can use a visual proof like we discussed earlier.

Dangers of example: Some things that may be true for some values may not always be true. For example, we may claim that a^2-b^2=2(a+b). If we say a=5 and b=3 and check, LHS=25-9=16, and RHS=2x8=16. But this is not always true! Check for a=4 and b=3.

Proofs and formulas: If we prove some relationship, that can then be used to calculate a useful quantity, then it becomes a *formula.*

For example, here is a famous proof for the sum of the first N natural numbers. [One can narrate the anecdote about Gauss finding this when in school he was asked to sum 1 to 100. It is a nice insight into how formulas as useful]

Let 1+2+3+...+N-1+N=x

Therefore, N+N-1+...+3+2+1=x

Adding term by term:
(1+N)+(2+N-1)+(3+N-2)+....+(N-2+3)+(N-1+2)+(N+1)=2x

Each term in the brackets is equal to N+1, and there are N such terms. Therefore,
$N(N+1)=2x$
or, x= $N(N+1)/2$

This relationship is going to hold true for all values of N. We can check it for any value, but the proof has already shown that it will be true. Along with being a proof, this is also a very useful thing that we often need, sometimes in doing other proofs. Hence, it becomes the *formula* for finding the sum of the first N natural numbers.

Let us consider something that we use in economics often. We have a concept called **expected value**. If you are going to get Rs 100 with a probability of ½ then the expected value is 1/2x100=50.

Let us consider the firm that has two options - high risk high reward, or low risk low reward. In high risk, like entering a new segment (think of reliance entering mobile comm with Jio), the firm will earn a profit of 100 crore each year, but the probability of failing is 0.5 i.e. every year there is a 50% chance that the firm closes down. The low risk strategy, continue in existing market, the firm will earn a profit of 25 crore each year and the probability of failing is only 0.1. Which strategy should the firm choose?

Let us consider the high risk strategy first. In the first year the firm gets 100. In the second year with a probability 0.5 it gets 100 and with a probability 0.5 it gets 0 and stops there. In the third year, which will happen only with probability 0.5, it again gets 100 with 0.5 prob i.e. total prob of

0.5x0.5… and so on.We need to calculate the lifetime expected profits, but the firm can theoretically go on till infinity!
$L_{HR}=100+100(0.5)+100(0.5)^2+100(0.5)^3+....$

Here we can deploy a very useful formula - sum of a geometric series
$1+x+x^2+x^3+...= 1/(1-x)$, if $x<1$

This formula comes from a very elegant proof
Let $1+x+x^2+x^3+...= a$
Then, $1+ax=1+x(1+x+x^2+x^3+...)=1+x+x^2+x^3+...$
Therefore, $a=1/(1-x)$

Question: Use the formula to find the lifetime expected profits of the two strategies

Then we can say
$L_{HR}=100(1+x+x^2+x^3+...)$, where $x=0.5$
$=100*1/(1-0.5)=200$

Similarly,
$L_{LR}=25(1+x+x^2+x^3+...)$, where $x=0.9$
$=25*1/(1-0.9)=250$

Hence, the firm should choose the low risk strategy.