Functions

Exponential functions

2^3=2x2x2

2^x=2x2x2… *x* times

Q. Can you explain why $a^x a^y = a^{x+y}$?

Q. Hence can you explain why $a^0=1$
Class exercise:

Organise into pairs
Person to my right is the depositor
Person to my left is the banker
The depositor chooses how much money to deposit in their bank account
The deposit is for 2 years and the interest rate is 10% per annum
The interest is calculated at the end of the year and added to the depositors bank account
The banker needs to tell the depositor how much money is there in their account at the end of the time period specified

Round 2 – now the roles are reversed
Person to my right is now the banker
Person to my left is now the depositor
The deposited sum, the time and the interest rates remain the same
But now, the interest is calculated at the end of every six months and added to the depositors bank account
The banker needs to tell the depositor how much money is there in their account at the end of the time period specified

Round 3 –roles are reversed again
Now, the interest is calculated at the end of every three months and added to the depositors bank account
The banker needs to tell the depositor how much money is there in their account at the end of the time period specified

Now both members together should come up with three graphs to show how the money in the account is changing with time for each of the three cases.

Eg. How much will an investment of Rs 100 grow in t years if the growth rate is 10% per year ?

$f(1)=100+0.1(100)=100(1+0.1)$
$f(2)=100(1+0.1)+0.1(100(1+0.1))= 100(1+0.1)(1+0.1)=100(1+0.1)^2$

Similarly, $f(t)=100(1+0.1)^t$

Suppose an amount Rs 100 is deposited in a savings account in a bank with annual interest rate 8%. The interest is paid at the end of the year.
The amount at the end of the year will grow to $100(1+.08)$.
Now, let the interest be calculated and paid after 6 months. Hence, after the first six months, the amount will be $100(1+0.04)$. And at the end of the year it will be $100(1+0.04)^2 = 100(1+0.08/2)^2$.
This is called **compounding**.
Let us approach this more generally
Initial amount = A
Annual interest rate = r
Number of compounding intervals = n (n=2 in the previous case)

Therefore amount at the end of the year = $A(1+r/n)^n$

What happens if we make n infinitely large?

Let us start with r=1 for simplicity.

As n becomes infinitely large, the expression converges to a constant value, which is called ‘e’.

n	$\left(1+\frac{1}{n}\right)^n$
1	2.0
2	2.25
4	2.4414
10	2.59374
100	2.704814
1,000	2.7169239
10,000	2.7181459
100,000	2.71826824
10,000,000	2.718281693

We write this using something called a *limit*, which we are going to see in a little more detail later:

$$e \equiv \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

By doing this we have moved from discrete compounding to continuous compounding, or continuous growth.

Now, what we want actually is $\lim_{n \to \infty} \left(1 + \frac{r}{n}\right)^n$

We can prove that it actually is equal to e^r.

Therefore in 1 year, an initial amount of A will become

$f(1)=Ae^r$

And like earlier, every year if will increase so that after t years it is

$f(t)=Ae^{rt}$

Here you can think of f(t) as deposit amount, GDP, prices, and r would be interest rate, growth rate and inflation respectively.

Q. Plot this using a spreadsheet and see what the function looks like for A =100, and different values of r (say 0.5, 5, and -1)

Q. What is its domain and range? (You can take one specific example and answer)

These measures of interest rate, growth rate, rate of inflation etc are continuous, and are different from the discrete or annual rates that are commonly used.

If the economy grows from 100 to 110 in 1 year, then the annual rate of growth is 10%.

[f(t+1)-f(t)]/f(t)=[110-100]/100

But does the economy grow discretely or continuously?

If we want to represent growth as a continuous process Ae^{rt} , then how do we find our growth rate r ?

$f(t)=Ae^{rt}$,
$f(t+1)= Ae^{r(t+1)}= Ae^{rt}(e^r)$

Therefore, $f(t+1)/f(t)= e^r$

In the earlier example, e^r = 110/100 = 1.1

From here how do we get to r?
We use logs!

Logarithmic functions

If $a^x = y$, then $\log_a(y)=x$

When we say $\log_a(y)$ we are asking the question $a^? = y$

Now we want to know $e^?=1.1$

r will be given by $\log_e(1.1) = 0.0953$ (appx)

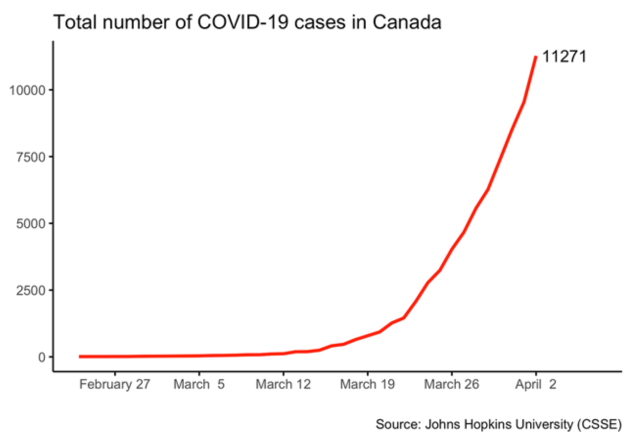
In economics we almost always use $\log_e(x)$ which is often written as just log(x), or sometimes as ln(x)

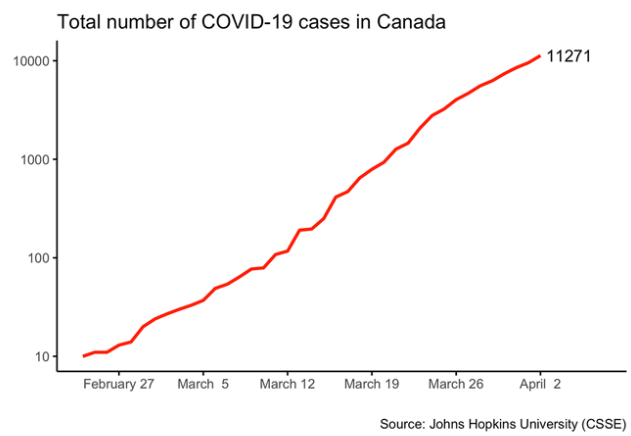
Q. Can you show that (i) log(ab)=log(a)+log(b), and (ii) log(a/b)=log(a)-log(b) ?

Therefore, r=log(f(t+1)/f(t))=log(f(t+1)) - log(f(t))

The graph of log(f(t)) vs t would be a straight line with r as the slope!! This is very useful and we can often interpret log-linear graphs using this.

For example, see below a log linear graph of covid cases in the first wave





Source

Q. Plot log(x) using a spreadsheet. What is its domain and range? Is it invertible? What is its inverse?