Functions of multiple variables

As we have said earlier, a function can have more than one independent variable. So, for a function with *n* independent variables, the input could be any ordered (i.e. we know what the first number represents, what the second number represents, and so on) combination of *n* numbers, or a vector of length *n.* The function is a rule that gives the value of the dependent variable – one real number – for each such possible combination.

For example, think of a production process for producing paper – you need capital (x_1), labour (x_2), electricity (x_3), raw material (x_4). The production function should tell us for each combination of these four inputs, how much paper (y) would be produced – multiple independent variables, one dependent variable. Hence the production function can be written as

$y=f(x_1, x_2, x_3, x_4)$

Note that now the domain has to be specified over all four independent variables.

Although functions with multiple independent variables can have any number of independent variables, most basic economic models deal with only two. This is because

(a) They are easy to represent on graphs (we will see how), and
(b) Most of the insights gained from such models are also applicable to functions with more than two independent variables

So, typically we would have a production function with only capital and labour as inputs. Or a utility function over only two goods.

Graphical representation

When we discussed functions with only one independent variable, we represented them on a 2-dimensional graph, with the horizontal axis being the independent variable and the vertical axis being the dependent variable. The function is plotted as a curve connecting all points with the coordinates (x,f(x)), i.e. for each value of the independent variable x in its domain, we plot the corresponding value of the dependent variable.

But now let us consider a function of two variables, say $z=f(x,y)=x^2+y^2$.

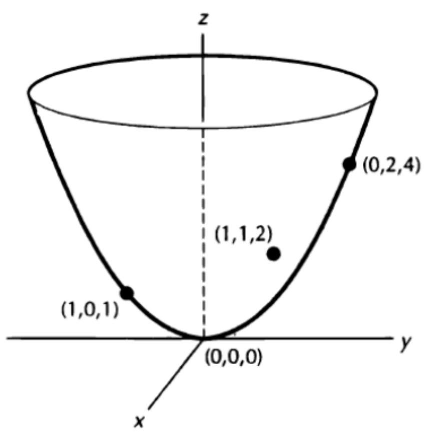
This is clearly a function where x and y can both be any real number. Hence the domain of the function is all combinations of x and y where each of them can be any real number. Since, x can be represented by a line, and y by another line - their combination can be represented by a plane. This is sometimes called the Real plane (as compared to the Real line), or $\mathbf{R}^2$ (as compared to **R**).

The domain is now the x-y plane, and the function f(x,y), gives us the value of the dependent variable, z, for every point on the plane. If we want to plot the value of z also, we need to use the third dimension available to us and draw a 3-D figure.

Typically we can use the computer to draw this, but we know the values that it can take.

Eg. f(0,0)=0, f(1,1)=?, f(1,0)=?, f(0,2)=?

Below is a figure which shows what the 3D graph would look like. (Note that the coordinates are in the form (x,y,z))



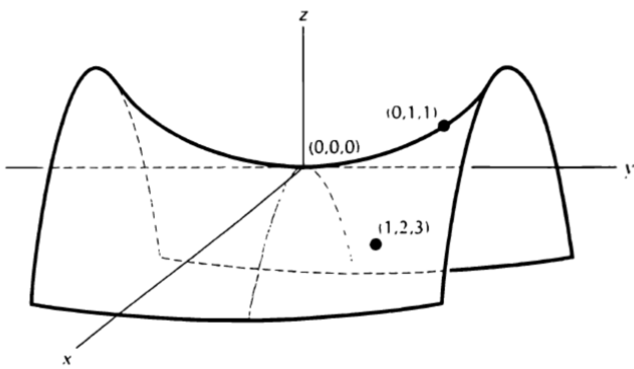
The graph of $f(x, y) = x^2 + y^2$.

Note that this is the 3-D version of a parabola, which is the graph of $f(x)=x^2$

Let us take another example:

$z=f(x,y)=y^2-x^2$

f(0,0)=?
f(0,1)=?
f(1,2)=?



The graph of $f(x, y) = y^2 - x^2$.

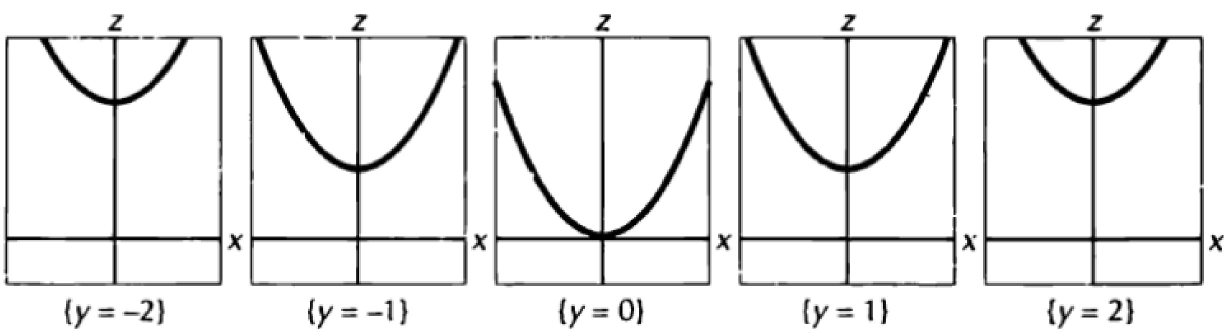
Let us go back to the first function

$z=f(x,y)=x^2+y^2$

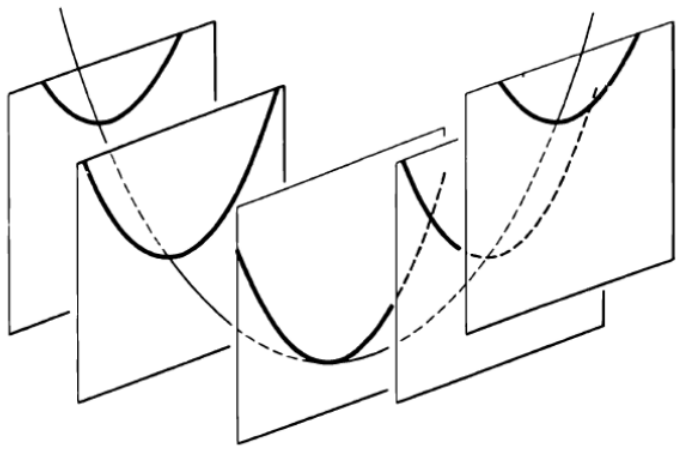
Let us think of this as a production function where x and y are the two inputs, let us say labour and capital. Now, we can fix the value of one input (let us say we fix capital y=10 in the short run), and only change the other. Then, the function becomes a single-variable function

$z=f(x,10)=x^2+100$

Hence, for different fixed values of y, we can have different single variable functions of x. Visually we can think of this as slicing the 3D graph in planes of y=10, y=5 etc.

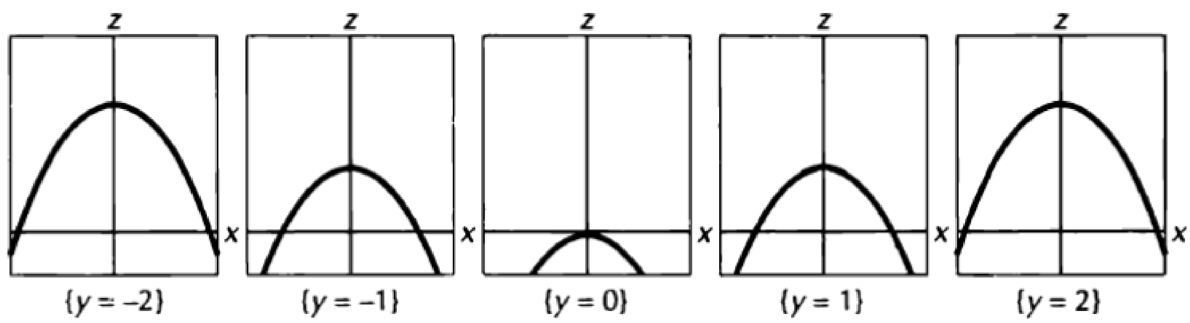


Slices of $z = x^2 + y^2$ ***in the*** $\{y = b\}$***-planes.***

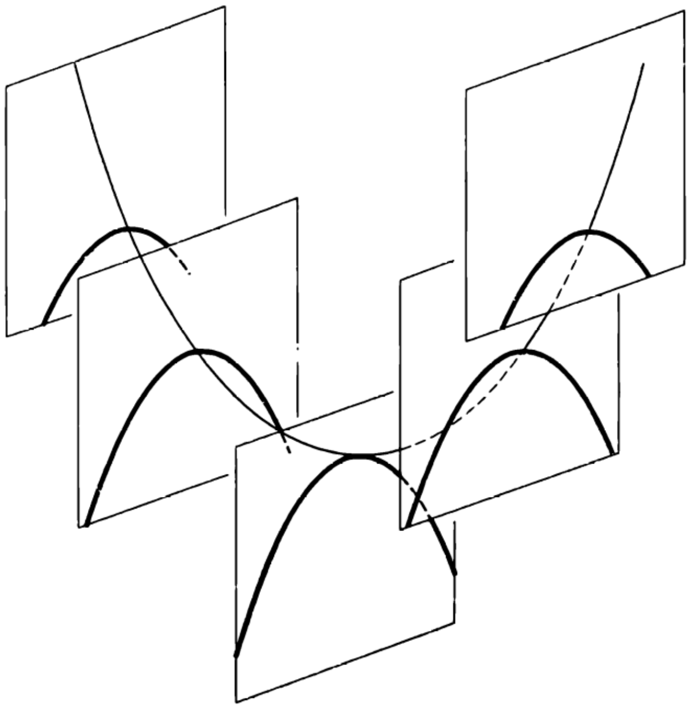


Putting the slices together.

Similarly for the other function



Restrictions of $z = y^2 - x^2$ to the planes $\{y = b\}$.



Putting the slices together.

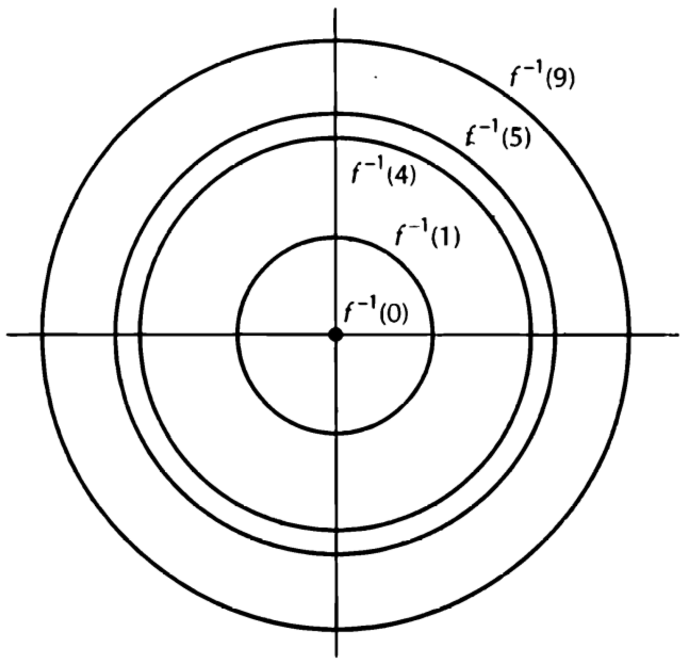
Draw the slices for the function
$f(x,y) = y - x$
$y=1$

Level curves

The level curve of a function f(x,y) at the value a, is the set of all {x,y}, where the function takes the value a.

For example, the level curve for the function $f(x,y)=x^2+y^2$ at the value 5 is given by all points (x,y) that satisfy the equation $x^2+y^2=5$

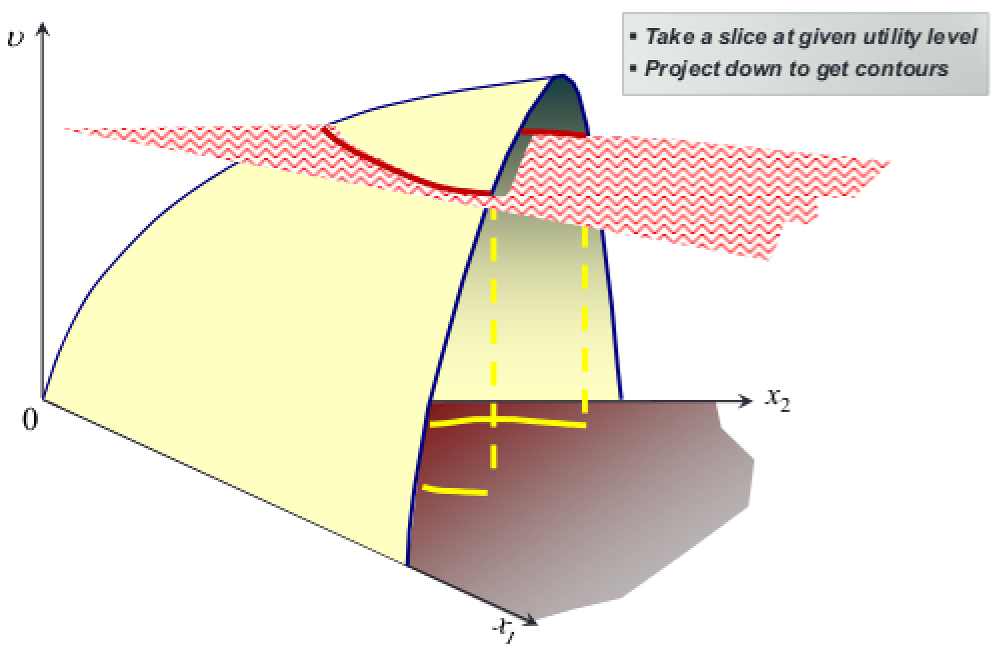
Hence, the level curve is drawn on the x-y plane itself. It contains all combinations of x and y that give a certain value of the dependent variable.

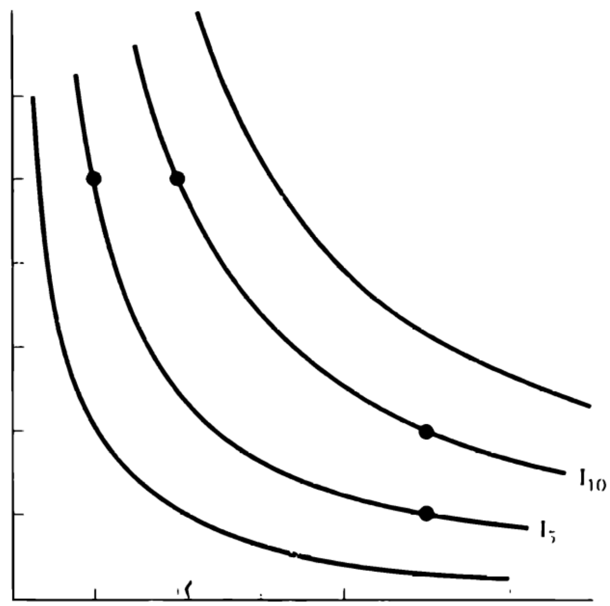


Level curves of $z = x^2 + y^2$.

Q. Have you seen something like this?

[Indifference curves, Isoquants]





Q. Draw indifference curves for the following utility function v=u(x,y), for the values v=4, and v=8.