Slope of a function
As we have previously discussed, the slope of a function is given by how much the dependent variable changes as a result of a unit change in the independent variable. It is a very useful concept in economics. When applied to a production function it gives us marginal product i.e. how much the output will increase if we increase the input by a unit. When applied to a cost function marginal cost i.e. how much the cost will increase if we increase production by 1 unit. And there are many other such applications.

We know that if the function is linear then we can identify the slope directly from looking at the function itself if it is in slope-intercept form (y=mx+c). Or if we have a graph of the function, we can take any two points (x1,y1) and (x2,y2) and the slope will be given by (y2-y1)/(x2-x1).

Q. Why is this the slope?
[If say y2-y1 is 10 and x2-x1 is 5. So, if x increases by 5, then y increases by 10. Using unitary method, if x increases by 1, y increases by 10/5.]

We can do this because the slope is the same at any point on a straight line. But what about non-linear functions? Let us take the example of $y=x^2$.

This could be the cost function of producing a continuous good, like digging a well of depth 'x'.

Let us say we select two points (0,0) and (2,4). The slope would be 4/2=2 [show it on the graph]. But if we select (1,1) and (2,4), the slope is 3!, And between (0,0) and (1,1) the slope is 1!! So the slope is different based on the two points we select.

Slope at a point
Since the slope of a non-linear function changes continuously, we would like to know the slope at any given point. For example, let us think of the cost function mentioned above. We would like to know at any depth 'x', say 1m, what is the marginal cost of digging more.

Graphically we can see that this will be the same as the slope of a straight line which is a tangent to the curve at that given point. So, if we draw a tangent to the curve of $f(x)=x^2$ at x=1, the slope of the line will be the slope of the curve.

Q. What is a tangent?

We can see this graphically but how do we obtain is algebraically. To do this we will have to use the idea of a 'limit'. We had encountered this before in the definition of the constant e. We will first see why we need it in this context and then see how we find the limit of a function, and then use that to find the slope of x^2.

We know that we can find the slope of a straight line passing through any two points on the curve. So, if we take any two points (x1,y1) and (x2, y2), then we can find the slope of the line through them as equal to (y2-y1)/(x2-x1). Let (x1,y1) be the point we want to find the slope at (for example (1,1)). Then as (x2,y2) moves closer to it, we can see that the line passing through the two points becomes closer to the tangent. And the line becomes the same as the tangent when the two points coincide.

To show this more explicitly, we write x2 as x1+a, where a is the difference between x1 and x2, i.e. a=x1-x2. Therefore, x2 becoming closer to x1 means a becomes close to zero.

The slope of the line passing through the two points is
$(y2-y1)/(x2-x1)=(x2^2-x1^2)/(x2-x1)=((x1+a)^2-x1^2)/a$

Therefore the slope of the line at (x1,y1) is $((x1+a)^2-x1^2)/a$ as a becomes close to zero. This is written in the form of a **limit**.

Obviously we cannot find its value by putting a=0 since anything divided by zero is not defined. But we can find what value it gets close to as a gets very close to zero.

For example let us take the function $g(x)=(x^2-4)/(x-2)$. We want to find the value of this function as x approaches 2. Again we cannot simply substitute x=2 as g(2) does not exist. In fact the domain of this function will not contain the point 2. Let us try and plot its graph on a spreadsheet.

The way we find the limit of this function is to use the fact that we are not actually saying that x=2 but that x is very close to 2. Hence x-2 is never zero and we can cancel it in the numerator and denominator.

Q. Can you use the same idea to find the limit to get the slope?

We find that the slope of the function $y=x^2$ at any point (x1,y1) is 2x1. Hence the slope itself is a function of x and is equal to 2x.

This process of finding the slope of a function is called **differentiation**, and the slope function is called a **derivative.**

The derivative is written as dy/dx, or as f'(x).

Q. Can you use the limit method to find the derivative of the function $y=h(x)=x^3$? (Note: you can use the formula $(a+b)^3=a^3+3a^2b+3ab^2+b^3$