Implicit functions

In many economic applications we will need to find the slope of a function that is not given to us in the form we are used to but in the form of an equation:

Eg. Isoquant

A production function (of two variables) is depicted as an isoquant by taking the 'level curve' at a particular value.

Eg. if the production function is $y=f(k,l)=kl$, then we can draw its isoquant $kl=5$

Q. What is the slope of the isoquant?
[MRTS]

How do we find it?

What we want to find is dk/dl along the isoquant but we do not know k as a function of l. So, let us write it like that
$l\,k(l)=5$

Product rule
$k(l) + l\,k'(l)=0$

$k'(l)= - k/l$

Note: We could also have written $k(l)=5/l$ and differentiated but doing it in this form will give us some useful intuition.

Q. What does it mean to say that the slope of the isoquant is $-(k/l)$?

[When labour is higher we need much smaller amount of capital to substitute one unit of labour]

Another example. Think of a log-linear graph of covid (or GDP). We can see that it follows a straight line but the dependent variable is not y but ln(y)

$\ln(y)=0.2t+5$

$\ln(y(t))=0.2t+5$

Using chain rule

$1/y\; y'(t)=0.2$

This means that with every increase in t, y increases by a factor of 0.2, or by 20%. So if y is 10, the rate of change of y is 2, and if y is 100 it is 20, and so on.

Increasing and decreasing functions

Q. What is the relationship of slope to whether a function is increasing or decreasing?
[+ve increasing, -ve decreasing]

Now, since we can find the slope of any function using differentiation, we can also see whether it is increasing or decreasing. Let us use the examples we did in the assignment.

(a) $f(x) = 4 - (x-2)^2$, for $x > 2$

(b) $$g(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ x - 1 & \text{if } 1 \leq x < 2 \\ 1 & \text{if } x \geq 2 \end{cases}$$

(c) $h(x) = e^{-2x}$

Find the slope of each of these functions by differentiating and then check whether it is increasing or decreasing.

Q. Let us take the first function and change its domain to be all real numbers. Now we can see that its slope changes from positive to negative. At what point is it 0? What is the significance of this point?

[x=2, it is where the function is maximum]

Second order derivatives

Let us consider two production functions $y = f(x) = x^2$, and $y = g(x) = \sqrt{x}$ (considering only the positive root)

Q. Are they increasing or decreasing?

[Both are increasing]

This makes sense because if you increase the input, say labour, you would expect the output to also increase.

Let us draw their graphs.

Q. What is a key difference between them that you notice?
[Increasing at an increasing rate, and at a decreasing rate]

Q. What does this mean?
[The marginal product is higher at higher levels of production in one case, and lower in another - discuss examples]

Hence, even though both have positive slope, it is decreasing in one case, and increasing in the other.

If the slope is increasing, it means that the slope of the slope is positive!

[Draw graphs to show]

The slope of the slope is called the second order derivative of the function. Itis denoted either as $f''(x)$ or as d^2y/dx^2

Functions with a positive second derivative are called **convex** and those with negative second derivatives are called **concave**.

[Do not confuse these with concave and convex mirrors and lenses from physics as all curves are physically both concave and convex from either side]

Therefore, the function $f(x) = x^2$, is increasing and convex for $x>0$

Q. Find the second derivative of $g(x) = \sqrt{x}$

It is negative, and hence the function $g(x) = \sqrt{x}$, is increasing and concave for $x>0$

For each of the following functions, check whether they are increasing/decreasing and convex/concave

(a) $x^3 - 2x + 2, \ x \geq 1$

(b) $x^3 - 2x + 2, \ 0 \leq x \leq 0.5$

(c) $e^x(1 - x), \ x \geq 0$

Q. Can you show what the shape of the graph of a decreasing but convex function is going to be?