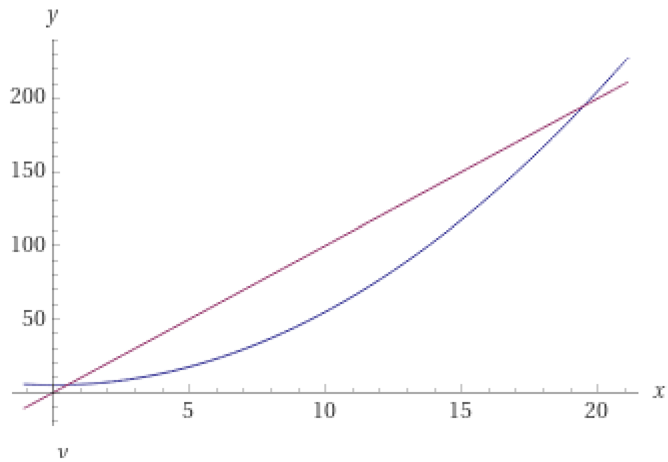
Let us take the example of digging up soil. The cost function is likely to be increasing and convex (why?). Let us assume it is given by c(x)= 0.5x^2 + 5

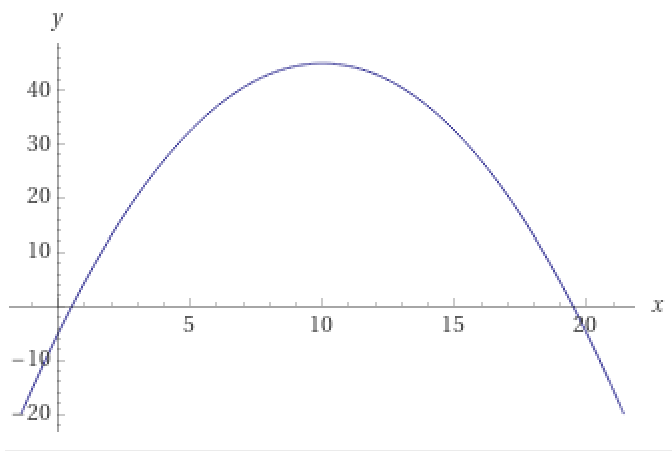
Now, let us assume that the soil dug up is sold at a constant price. Hence the revenue is given by a linear function r(x)=10x

Let us draw the graphs and see what it looks like.



Q. What is the profit in this graph? Where is it positive? Where is it negative? Where is it maximum?

The maximum point of the profit is more clearly seen if we directly draw the profit function f(x)= r(x)-c(x).



The maximum profit is achieved at the ‘crest’ of the curve or where the inverted U shape is.

This is called a ‘local’ maximum and similarly one can think of a local minimum for another function.

Local versus global maxima and minima

Critical points at extremes of the domain and where slope=0

For interior critical points:

First order condition: $f'(x)=0$

Second order condition:
$f''(x)<0$ -> maximum
$f''(x)>0$ -> minimum
$f''(x)=0$ -> indeterminate

Eg. Find the local minima/maxima of $x^2 - 4x + 1$

F.O.C.:
$f'(x)=2x-4=0$, $x=0.5$ There is only one critical point

S.O.C.
$f''(x)=2>0$. Hence this is a minimum.

Q. A monopolist faces a demand curve

$p=10-4q$

The marginal cost is 2

What quantity should the monopolist produce to maximise profit

Ans:
Cost: $c(q)=2q$
Revenue: $r(q) =pq =(10-4q)q$

Profit: $f(q)=r(q)-c(q)=10q-4q^2-2q=8q-4q^2$

First Order Condition
F.O.C:
$f'(q)=0$
$\Rightarrow 8-8q=0$
$\Rightarrow q=1$

Therefore, there is a critical point at q=1 let us call this q*

Second Order Condition:
S.O.C.:
We find f''(q) at q=q*

f''(q)=-8

Therefore f''(q)<0 at q=q* (in this case it is negative for all values of q), therefore q=1 is a local maximum. As this is the only critical point, this is also the global maximum.

Hence, the maximum profit is attained at q=1, and the value of the maximum profit is f(1)=8-4=4

Let us do a few examples of finding maxima/minima with FOC and SOC

1. $e^{2x} - 5e^{x} + 4$
2. $e^{(x-1)} - x$
3. x^3-2x^2+x-5 [Here local max and min are not global]
4. $x^3-6x^2+12x-3$ [SOC is indeterminate]

Let us do another problem:

A firm's production function is

$$Q(L) = 12L^2 - \frac{1}{20}L^3$$

where L denotes the number of workers with $L \epsilon (0, 200)$.

(a) Find the size of workforce L^* that maximises output $Q(L)$.

(b) Write down an expression for output per worker and find the size of the workforce L^{**} that maximises this.

Exercise:

- A person has a total of 16 waking hours which she can spend on two possible activities. She spends h_1 hours on the first activity - working on her own, and h_2 hours on the second activity - working for a wage. Her income from the first activity is $300\sqrt{h_1}$. The wage she earns from the second activity is Rs 100 per hour. If she wants to maximise her income how should she ideally divide her time?

[Show substitution, FOC and SOC]

- A person has Rs 2000 to spend on buying milk and sugar. The price of milk is Rs 20 per unit and of sugar is Rs 40 per unit. The utility function that the individual is maximising is

$u(x,y)=\sqrt{xy}$, where x is the amount of milk and y is the amount of sugar. What is the optimal amount of milk and sugar that they can buy?