Integration

Let us consider a worker working with a salary of Rs 5 lakhs per year. Write down the total income earned by the worker as a function of time(measured in years)?

Draw graphs of the salary, and the income, against time.

Q.What is the relationship between the income graph and the salary graph?

Ans [Income is the area under salary : Income = salary x time]

Now consider the following salary structure: initially the salary is Rs 4 lakhs per year, then every five years the salary increases by 1 lakhs per year. Write down the salary as function of time and then the income as a function of time. And finally draw the graphs.

Q. Can you say what the relationship of the first graph to the second is?

Ans [The first is the slope of the second: Salary = d Income / d time]

Now consider a salary structure where the salary increases *continuously* as a function of time : $s=2t$

Draw the graph.

What would be the income function and graph? It is not very obvious. If salary was a non-linear function, say $s=t^2$, it is even harder to figure out the income function or graph.

Riemann integrals

One approximate way to do this is to convert it into the second case. Consider small chunks of time, Δt, and consider the function to be constant over this time.

Use a graph paper and find the Income at times 0, 2, 4.

A. Approx 0, 4, 16

But this is approximate and not efficient.

We can use the fact if we want to find the area under the curve of any function f(t), in this case the function is salary, and the area under the curve is income, then we need to find that function say g(t) such that f(t) is the slope of g(t).

Or, we want to find g(t) such that $f(t)=g'(t)$

Q. Now, if f(t) is 0.5t, then what is g(t)?

A. $g(t) = t^2$ + constant

Q. Why is there a constant?

Here g(t) is called the *antiderivative* of f(t). It is also called the **indefinite integral** of f(t).

Now, to find the area under the curve f(t), we need to specify two points between which we want to find the area. In this case, we start from t=0, to , let us say, t=2. Therefore the area would be $g(2)-g(0)=2^2-0^2=4$. As you can see we got the approximate answer from the Riemann integral.

Now let us introduce some notation:

The indefinite integral of f(x), which basically asks the question: which is that function whose slope is f(x), is written as

$\int f(x)dx$

Therefore we can say that $d/dx(\int f(x)dx) = f(x)$

A few simple rules can be derived from the rules of differentiation:
- $\int x^n dx= x^{n+1}/n+1 +C, n\neq-1$
- $\int e^{ax}dx= e^{ax} /a +C$
- $\int 1/x \, dx= \ln(x) + C$

Also,
$\int a\, f(x)dx = a \int f(x)dx$
$\int [f(x) + g(x)]dx= \int f(x)dx + \int g(x)dx$

Using these find the indefinite integrals of the following functions:
1. $3x^2 + 5x + 2$
2. $3/x - 8e^{-4x}$
3. $(x^3-3x+4)/x$
4. $(x-2)^2/\sqrt{x}$

Next, we look at how we found the area under a curve by calculating the indefinite integral at two points and taking the difference. This is called the **definite integral.**

So, if $g(x) = \int f(x)dx$,
Then g(a)-g(b) is written as $\int_b^a f(x)dx$

So, the area that we found was $\int_0^2 2x\, dx = [x^2]_0^2=2^2-0^2=4$

Note that the definite integral is not a function of x but of its limits i.e. a and b.

Exercise: Find the area under the curve of the given functions for the intervals given

1. x^3 , over [0,1]
2. $3x^2$, over [0,2]
3. e^x, over [-1,1]
4. $1/x^2$, over [1,10]

The area between the demand curve and the price line is called the consumer surplus. The area between the price line and the supply curve is called the producer surplus.
Let the demand curve be given by P=d(Q)=50-0.1Q
Let the supply curve be given by P=s(Q)=20+0.2Q

Find the equilibrium price.
Then find the consumer and producer surplus