

Vectors

So far in these course we have seen:

Numbers: Natural numbers, Integers, Real numbers, and operations on them

Relationships between numbers, and finding unknown numbers: Equations and inequalities

Rules about relationships between independent and dependent quantities: Functions - finding slopes of functions, and finding the area under functions

Now we are going to look at structures or concepts that help us deal with a collection of numbers.

The first such concept is called a **vector**. We have already come across this word a few times in this course but now we will proceed to understand it in more detail.

A vector is an ordered set of numbers. For example if we know the amount of rainfall in Sarjapur over all 31 days of October in mm as 0,10,12,0,0,..etc., then we can call it a vector written as (0, 10, 13,0,0,...). Each day's rainfall is an **element** of the vector. This particular vector is going to have 31 elements. The number of elements is also referred to as the **length** of the vector. [Note that when we express intervals, for example in domain and range of functions, we also use parentheses i.e. (), but the meaning is different here]

In general we will denote vectors with a letter, like \mathbf{x} , and its elements with x_i , $i=1,2,3,\dots$ and the vector itself will be (x_1, x_2, x_3,\dots) .

We can graphically denote a vector as a point in n-dimensional space. For example, we already denote a vector of length two on a two dimensional graph. We can visualise up to 3 dimensions. Higher dimensional graphs theoretically exist although it is not possible to visualise them.

There is no new information in writing out numbers as a vector, but the form allows us to define some operations that are frequently used, and then get some useful results. [This again relates to the language of maths].

So, let us define operations on vectors:

- 1) Addition (or subtraction) of two vectors is possible only if they have the same number of elements i.e. they are of the same length. For example, if vector \mathbf{a} denotes the amounts of money (in rupees lakhs) spent by a firm on its inputs in year 2020 - labour, capital, electricity, raw materials, eg. (50, 20, 10,70) and \mathbf{b} denotes the same thing in 2021, eg. (40, 30, 20, 80), then $\mathbf{c}=\mathbf{a}+\mathbf{b}$, will be element-wise addition of the two vectors to give the total sum spent on each input in the two years combined i.e. (90,50,30,150)
- 2) The product of a vector multiplied (or divided) by a constant gives a new vector where each element is multiplied (or divided by the constant). So, the average amount spent on each input over the two years is given by $\mathbf{d}=(\mathbf{a}+\mathbf{b})/2 = (45, 25, 15, 75)$

- 3) Two vectors can be multiplied only if they have the same number of elements, i.e. they are of the same length. It is possible to define an element-by-element multiplication that gives a new vector of the same length, but a more useful concept is called an **inner product** that does the element-by-element multiplication and then adds them up. (just doing element wise multiplication to get a new vector is called a Hadamard product)

The inner product is also called a dot product, because it is represented by a dot.

$$\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^n a_i b_i$$

This is very useful, especially when asking a computer to perform complex operations. When you learn coding next semester, you will learn to define and use vectors for many applications.

For example, if a course has four assessments with weightages given by a vector $\mathbf{w}=(0.2,0.1,0.4,0.3)$. Then if we know any students scores in the four assessments, for example $\mathbf{s}=(60\%,70\%,80\%,60\%)$, then the students overall score in the course is given by $\mathbf{w} \cdot \mathbf{s}=(0.2 \times 60\%)+(0.1 \times 70\%)+(0.4 \times 80\%)+(0.3 \times 60\%)=69\%$

Using this one can easily write a computer program where any student can input their assessment scores and find out their overall score, even for a course with 20 assessments. In fact, this is what Moodle does!

Note that for a dot product the order does not matter,
 $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$

We can prove this property as follows:

$$\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^n a_i b_i = \sum_{i=1}^n b_i a_i = \mathbf{b} \cdot \mathbf{a}$$

Also, we can open brackets while taking dot product (also called the distributive property)i.e

$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$$

Q. Can you prove this property using this definition of dot product?

A.
$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \sum_{i=1}^n a_i (b_i + c_i) = \sum_{i=1}^n a_i b_i + \sum_{i=1}^n a_i c_i = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$$

Q. Suppose a catering service provides three different types of meals A, B and C. The price of each type of meal is given by the price vector \mathbf{p} . The number of meal of each type sold in a day

is given by the vector \mathbf{m} . Show that the dot product $\mathbf{p} \cdot \mathbf{m}$ gives the revenue earned during the day.

Q. Suppose there are n goods in an economy. The price of each good is given by the price vector \mathbf{p} . A particular firm can produce one or more types of goods. Its output is given by the vector \mathbf{q} . Its inputs are also going to be some goods in the same economy. Hence, its inputs are given by the vector \mathbf{x} . Show that the profit earned by any firm is given by $\mathbf{p} \cdot (\mathbf{q} - \mathbf{x})$.

Let us imagine a prediction game. Suppose the scores of Virat Kohli in n consecutive matches (or the price of a stock on n consecutive dates) is given by a vector \mathbf{x} of length n . I tried to predict the scores before each match - my predictions are given by \mathbf{a} also a vector of length n . My friend also tried to predict the score. His predictions are given by \mathbf{b} . Once we know \mathbf{x} , \mathbf{a} and \mathbf{b} , how do we check which one of us predicted better?

You may want to compare averages. But if the averages were the same, does that mean both of us predicted equally well? You can think of an example where one person got it exactly right every time while the other over and underpredicted alternatively, leading to the same average.

You can perhaps think of errors and who makes the lowest errors. But errors can be positive and negative. Hence we should think of either absolute or square.

We can define the error vectors \mathbf{e}_a and \mathbf{e}_b as $\mathbf{a} - \mathbf{x}$ and $\mathbf{b} - \mathbf{x}$ respectively. Now we can compare who made more errors by taking the sum of the squares (or absolute values) of each element.

The one who has a larger sum of squared elements has made larger errors overall and hence did a worse job of predicting.

This idea that the size of the vector is obtained by summing over squares of the elements is called the norm (or more precisely the Euclidean norm) of the vector. More precisely

$$\|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2 + \dots} = \sqrt{\sum_{i=1}^n a_i^2}$$

Also note that $\|\mathbf{a}\|^2 = \mathbf{a} \cdot \mathbf{a}$

This quantity also has a geometrical meaning. Consider a two dimensional vector $\mathbf{x} = (1, 2)$. We know that we can plot this as a point on a graph with axes x_1 and x_2 at $x_1=1$, and $x_2=2$.

Q. What would be the meaning of the norm here?

A. Using Pythagoras' theorem, the norm would be the distance from the origin to the point.

This makes sense. If we wanted to see how far the errors are from 0, the length of a line connecting the origin to the point denoting the error vector should tell us that. Hence, vectors are also represented as a directed line (sometimes with an arrow) of length $\|\mathbf{x}\|$.

We can visualise this in 2D and 3D space, but it continues to hold true in n-D space as well.

Graphical representation of vectors (only if time permits)

Graphical interpretation of addition, subtraction and scalar multiplication

Orthogonality and its equivalence to $\mathbf{a} \cdot \mathbf{b} = 0$

Q. For given vector \mathbf{x} , what is the graphical representation of $-0.5\mathbf{x}$.

Q. Prove using vectors that two sides of a triangle add up to more than the third i.e.

$$\|\mathbf{a}\| + \|\mathbf{b}\| > \|\mathbf{a} + \mathbf{b}\|$$