Can we divide by a matrix?

Instead of dividing, we will use the concept of multiplication already defined and replace dividing by **A** with multiplying by the inverse of **A.**

If we think in terms of number, y/x can be written as y(1/x). Here 1/x is the reciprocal of x, which means when it is multiplied by x it gives 1.

We will use the same analogy in matrices, except we will not call it a reciprocal but an **inverse**. It is important to not confuse this with the inverse of a function (although there are some similarities when matrices are used as functions).

An inverse of a matrix **A** is written as $\mathbf{A}^{-1}$ and is defined as that matrix which when multiplied by **A** gives the identity matrix i.e $\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$.

Now let us look at the system of linear equations:

$\mathbf{Ax=b}$

If we multiply both sides with $\mathbf{A}^{-1}$, we get

$\mathbf{A}^{-1}\mathbf{Ax}=\mathbf{A}^{-1}\mathbf{b}$
$\mathbf{Ix}=\mathbf{A}^{-1}\mathbf{b}$
$\mathbf{x}=\mathbf{A}^{-1}\mathbf{b}$

Therefore, if we can find the inverse of the matrix **A**, then we can find the unknown vector **x**.

Q. Can you show that for the matrix **A** to be invertible i.e. for its inverse to exist, it has to be a square matrix? [Assume it is not square and show that the definition cannot apply]

This itself gives us an important insight into the system of linear equations. For the solution **x** to exist, $\mathbf{A}^{-1}$ has to exist, and for $\mathbf{A}^{-1}$ to exist, **A** has to be a square matrix.

Q. What does **A** being a square matrix imply for the system of linear equations?
A.[The number of equations has to be equal to the number of unknowns]

Next, let us try to find the inverse of a 2x2 matrix

Let **A**=	3	4
	1	2
Let $\mathbf{A}^{-1}$ =	w	x
	y	z

Therefore, we can write down the matrix $\mathbf{AA}^{-1}$

$$= \begin{matrix} 3w+4y & 3x+4z \\ w+2y & x+2z \end{matrix}$$

For this to be an identity matrix, it has to be equal to $\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}$

Therefore,
$3w+4y=1$
$3x+4z=0$
$w+2y=0$
$x+2z=1$

We can solve these to get
$w=1$
$y=-1/2$
$x=-2$
$z=3/2$

Instead of numbers, let us keep elements of the matrix **A** in the form a_{ij}

Therefore we need to find w,x,y,z such that it is the inverse of $\begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{matrix}$

We can write $\mathbf{AA}^{-1}$ as

$$= \begin{matrix} a_{11}+a_{12}y & a_{11}x+a_{12}z \\ a_{21}+a_{22}y & a_{21}x+a_{22}z \end{matrix}$$

The equations now become

$a_{11}w+a_{12}y=1$
$a_{11}x+a_{12}z=0$
$a_{21}w+a_{22}y=0$
$a_{21}x+a_{22}z=1$

Solving these, we get
$w=a_{22}/(a_{11}a_{22} - a_{12}a_{21})$
$x=-a_{12}/(a_{11}a_{22} - a_{12}a_{21})$
$y=-a_{21}/(a_{11}a_{22} - a_{12}a_{21})$
$z=a_{11}/(a_{11}a_{22} - a_{12}a_{21})$

Or, the inverse of **A** is

$$1/(a_{11}a_{22} - a_{12}a_{21}) \quad \begin{matrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{matrix}$$

This number ($a_{11}a_{22} - a_{12}a_{21}$) that each element of the matrix is getting divided by is called the **determinant** of the matrix, represented as |**A**|

Note that if |A|=0, then the inverse of the matrix does not exist.

Therefore, for the solution of a system of equations **Ax=b** given by **x=A**$^{-1}$**b** to exist, we need the inverse of **A** to exist, for which we need **A** to be a square matrix and for the determinant of **A** to be non-zero**.**

Let us try to find the solution to the exam question using this

4x-2y=-4
7x-3y=-4

Here **A**= $\begin{matrix} 4 & -2 \\ 7 & -3 \end{matrix}$

b = $\begin{matrix} -4 \\ -4 \end{matrix}$

For calculating **A**$^{-1}$, we find the determinant of **A =** -12 - (-14)=2 . This is not zero, so a solution exists

Therefore **A**$^{-1}$ = $\begin{matrix} -3/2 & 2/2 \\ -7/2 & 4/2 \end{matrix}$

Therefore **x** = **A**$^{-1}$**b**= $\begin{matrix} 2 \\ 6 \end{matrix}$

Exercise: Solve the following by using the inverse

$$\begin{aligned} 2x + y &= 3 \\ 2x + 2y &= 4 \end{aligned}$$

This method of solving linear equations is called Cramer's rule. The advantage of this is that it can be programmed into a computer as a step-by-step code and there is no need of cognitive judgement of whether to use elimination method or substitution method etc.

Also, this can be extended to any number of variables. The process of finding inverse of matrices of size 3 or more is longer but it is still a formula that can be coded in a computer.

Let us now go back to our problem with the catering service.
20p1 + 40p2 +50p3 = 4000
35p1 + 30p2 +45p3 = 3900

Q. Can we find the solution here?

A. No, we need one more equation.

Let the third equation be

10p1 + 30p2 +20p3 = 2150

Use this website to find the inverse of the matrix and find the prices.

Ans: p1=30, p2=35, p3=40

Application: Leontief model

Consider a sector of the economy, say Power sector, which generated electricity. Now let us take the example of a sector that uses power, say steel. Let us now say that 0.5 units of power are required to produced one unit of steel. If we assume a linear production function, then we can say that if X unit of steel are produced then 0.5 X units of power are consumed. In this way if we can account for all sectors in the economy that consume power, we can account for all the units of power that have been exchanged in the economy. And then we can do this for every sector. This is called the input-output model or a Leontief model of the economy.

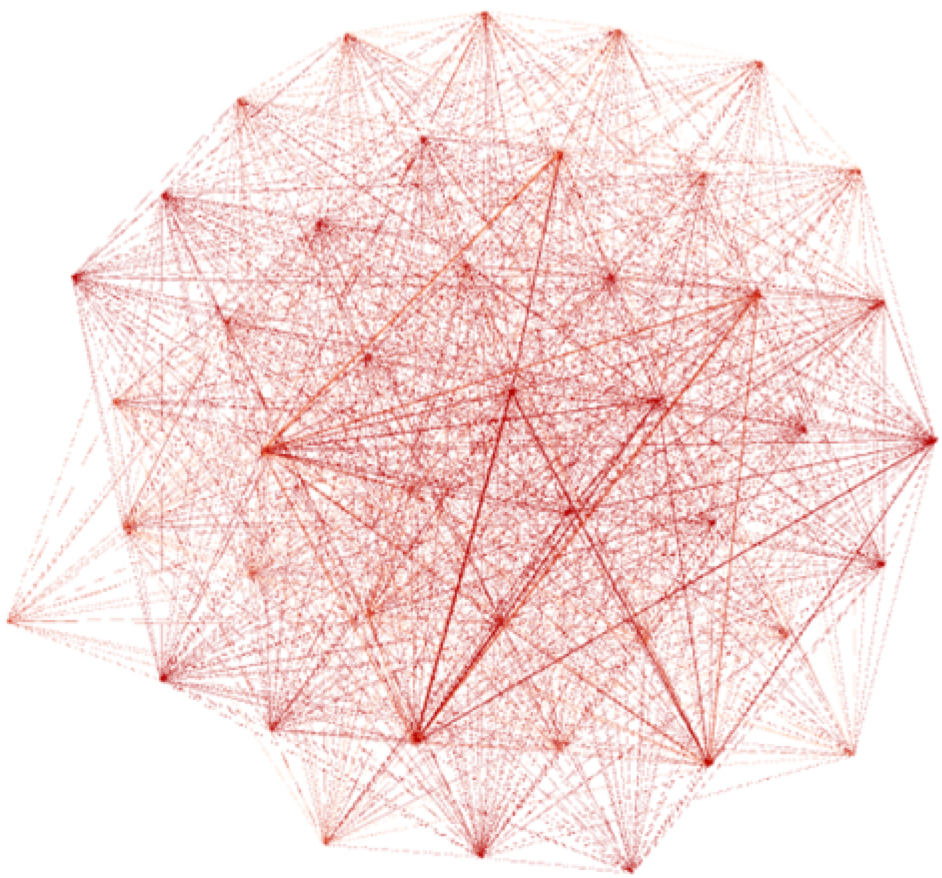
Let x_i represent the total output of sector i of the economy. Let a_{ij} represent the number of units of good i used to produce 1 unit of good j. Therefore the number of units of good i that were consumed by sector j are a_{ij} x_j. This represents the flow of goods or services from sector i to sector j. So, if we add up all the flow of good i to all the sectors of the economy, we get the total output of good i that is consumed by all other sectors.

This is given by

$a_{i1}x_1+a_{i2}x_2+a_{i3}x_3+....+a_{i4}x_4$

And we can do this for every sector, thus giving us a nxn matrix A whose every element a_{ij} gives how much of sector i is used as input by sector j to produce its output. This is called the input-output matrix.

Now, if you remember the matrix representation of a social network, we can also represent the economy as a network of sectors with the links being not 1s and 0s, but numbers that represent how intensively one sector's goods is used in the production of another sector.



Source: *Interconnected: The economy as a network. Krithika Raghavan, 2018.*

And then we can ask similar questions about which are the most important sectors of the economy.

Also, we can write the flows of goods and services as a system of linear equations.

Remember that the output of sector i that gets consumed as intermediate input in all other sectors is given by

$a_{i1}x_1+a_{i2}x_2+a_{i3}x_3+....+a_{i4}x_4$

The total output of sector i would be this along with the direct consumption. For example part of the output of the power sector gets consumed by other industries, but a part of it is also directly consumed by consumers. This final demand of good i is denoted as b_i.

Therefore

$x_i = a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 + ... + a_{i4}x_4 + b_i$

We will have n such equations which can be written as

x=Ax+b

This model was formulated in the days of central planning where they had to decide how much each sector should produce. The idea was that if we know the final demand **b** and the production functions, which will give us the input-output matrix **A,** then we can find out how much each sector should produce.

(I-A)x=b
x=(I-A)$^{-1}$b

Let us do an example:

Consider an economy divided into an agricultural sector (A) and an industrial sector (I). To produce one unit in sector A requires $1/6$ unit from A and $1/4$ unit from I. To produce one unit in sector I requires $1/4$ unit from A and $1/4$ unit from I. Suppose final demands in each of the two sectors are 60 units.

(a) Write down the Leontief system for this economy.

(b) Find the number of units that have to be produced in each sector in order to meet the final demands.